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The Brinkman model for natural convection in a porous layer: effects of nonuniform thermal gradient

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Abstract—The effects of nonlinear temperature distribution on stability and natural convection in a horizontal porous layer, with heating from below, are investigated using the Brinkman model. The horizontal boundaries are either rigid/rigid, rigid/stress-free, or stress-free/stress-free. Constant-flux thermal boundary conditions are considered for which the onset of convection is known to correspond to a vanishingly small wavenumber. An analytical solution for the flow and heat transfer variables, based on a parallel flow assumption, is obtained in terms of the Darcy–Rayleigh number, R, and the Darcy number, Da. The critical Rayleigh number for the onset of convection arising from sudden heating or cooling at the boundaries is also predicted. Various basic temperature profiles are considered. Closed form solutions are obtained from which results for a viscous fluid ($Da \to \infty$) and the Darcy porous medium ($Da \to 0$) emerge from the present analysis as limiting cases.

INTRODUCTION

NATURAL convection in a porous medium limited by two parallel plates maintaining an adverse temperature gradient is of interest to physicists, geophysicists and engineers. Applications include convection in the earth's crust, underground spread of pollutants, high performance insulation development and geothermal energy extraction. A large cross section of the fundamental research on this topic has been reviewed by Cheng [1].

The early work by Lapwood [2] determined the conditions for the onset of convection in a porous medium with horizontal isothermal boundaries. On the basis of a linear stability analysis it was found that convection occurs at Rayleigh numbers above $4\pi^2$. This result has been confirmed experimentally by Schneider [3], Katto and Masuoka [4], Combarnous [5], Bories [6], and Close *et al.* [7]. The stability of convection in a horizontal porous layer, subjected to an inclined temperature gradient of finite amplitude, was investigated by Weber [8] and Nield [9] respectively. The results showed that the critical Rayleigh number is always higher than $4\pi^2$. The onset of natural convection in a porous layer under other boundary conditions has been discussed by Nield [10] and Ribando and Torrance [11]. The stability of horizontal porous and viscous fluid layer, when the thermal gradient is not uniform, has been considered by Nield [12]. Using a Galerkin method, critical Rayleigh numbers were predicted by this author for various nonlinear basic temperature distributions and constant-flux conditions at both horizontal boundaries.

Most of the work on the onset of convection in a porous medium is based on Darcy's law which only takes into account the friction offered by the solid particles to the fluid. This is found to give satisfactory results when the porous medium is closely packed and with low porosity. On the other hand, Darcy's law cannot account for the no-slip boundary condition at the interface of a porous medium and a solid boundary, nor satisfy the continuity of velocity at the interface of a porous medium in contact with a viscous fluid. The Brinkman [13] extension of Darcy's law gets around these obstacles by adding a viscous like term to the equations.

The Brinkman model was used by Walker and Homsy [14] to determine the critical Rayleigh number against Darcy number for the case of conducting no-slip boundaries. Rudraiah *et al.* [15] have considered the Brinkman equation to study the onset of convection with nonlinear basic temperature profiles. The resulting critical Rayleigh numbers, obtained by using a one-term Galerkin approximation in their calculation, were found to be in good agreement, in the limit of very large Darcy numbers, with the values reported by Nield [12] for the viscous-fluid problem. However, in the limit of very small Darcy numbers, a large discrepancy was observed between their results and those obtained by Nield [12] on the basis of the Darcy equations. Recently, the Brinkman model was used by Vasseur *et al.* [16] to study the onset of convection in a porous medium heated from below by a constant heat flux. Using a parallel flow approximation critical Rayleigh numbers were obtained explicitly in terms of the Darcy number for various hydrodynamic boundary conditions. It was found that the results of viscous fluid and the Darcy medium emerge from their solution as special cases. It is thus evident that the discrepancies observed in the results obtained by Rudraiah *et al.* [15] are due to

NOMENCLATURE

- A aspect ratio, H'/L'
- C dimensionless temperature gradient in x direction
- Da Darcy number, K/L'^2
- g gravitational acceleration
- H' depth of cavity
- k effective thermal conductivity of the saturated porous medium
- K permeability of the porous medium
- L' width of cavity
- Nu Nusselt number, equation (17)
- q' uniform heat flux
- R Darcy–Rayleigh number, $g\beta KL'^2q'/k\alpha_f\nu$
- Ra Rayleigh number for a fluid, R/Da
- R_c critical Rayleigh number for a porous medium
- Ra_c critical Rayleigh number for a fluid
- S dimensionless uniform heat sink
- t dimensionless time
- T^* dimensionless temperature
- T dimensionless quasi-steady state temperature, T^*-St/σ
- $T_r^{*\prime}$ reference temperature at the origin
- $\Delta T'$ temperature scale, $q'L'/k$
- ΔT wall to wall dimensionless temperature difference at $x = 0$
- u dimensionless velocity in x-direction, $u'L'/\alpha_f$
- v dimensionless velocity in y-direction, $v'L'/\alpha_f$
- x dimensionless horizontal coordinate, x'/L'
- y dimensionless vertical velocity, y'/L'.

Greek symbols

- α dimensionless Darcy parameter, $Da^{-1/2}$
- α_f thermal diffusivity of fluid
- β coefficient of thermal expansion of the fluid
- θ y-dependent temperature term
- ν kinematic viscosity
- ρ density of fluid
- $(\rho C)_f$ heat capacity of fluid
- $(\rho C)_p$ heat capacity of saturated porous medium
- σ heat capacity ratio, $(\rho C)_p/(\rho C)_f$
- ψ dimensionless stream function, ψ'/α_f.

Superscript

- $'$ dimensional quantity.

the use of a single-term Galerkin expansion in their calculation.

The objective of the present study is to determine the critical Rayleigh numbers for the onset of convection, in the case of the Brinkman model, using the parallel flow analysis. In the first part of this paper, a shallow porous cavity heated from below by a constant heat flux, the other surfaces being insulated, is considered with various hydrodynamic boundary conditions on the upper and lower surfaces. Critical Rayleigh numbers covering the whole range of Darcy numbers, from the limit cases of Darcy to viscous fluid, are obtained. In the second part of this paper, the problem of Rudraiah *et al.* [15], namely the onset of convection in a Brinkman layer with non-linear basic temperature profiles, will be reconsidered using the parallel flow analysis. It is demonstrated that the resulting solution, in contradiction with that of Rudraiah *et al.* [15], can recover the limiting case of a pure Darcy layer.

MATHEMATICAL FORMULATION AND ANALYTICAL SOLUTION

We consider a fluid-saturated porous layer contained in a horizontal rectangular cavity of elongated shape, bounded by two rigid vertical side walls and two long horizontal boundaries at $y' = 0$ and L' that may be both rigid, upper stress-free and lower rigid, or both stress-free. The layer is heated from the bottom by a constant heat flux q' such that

$$q' = -k\frac{\partial T^{*\prime}}{\partial y'} \tag{1}$$

where k is the effective thermal conductivity of the saturated porous medium, $T^{*\prime}$, the temperature and primes denote dimensional variables. The other surfaces of the porous medium are insulated.

The following dimensionless variables are used

$$(x,y) = \frac{(x',y')}{L'}, \quad (u,v) = \frac{(u',v')}{\alpha_f/L'}$$

$$T^* = \frac{(T^{*\prime}-T_r^{*\prime})}{\Delta T'}, \quad \Delta T' = \frac{q'L'}{k}, \quad \psi = \frac{\psi'}{\alpha_f} \tag{2}$$

where (u', v'), ψ' and α_f represent the volume-averaged fluid velocity components, the stream function and the thermal diffusivity of the fluid, respectively. $T_r^{*\prime}$ is the temperature at the origin of the coordinate system and $\Delta T'$, a characteristic temperature difference.

Assuming the validity of Brinkman's law and the Boussinesq approximation and neglecting inertial effects, the equations describing conservation of momentum and energy in the medium are respectively:

$$\nabla^2\psi = Da\nabla^4\psi - R\frac{\partial T^*}{\partial x} \quad (3)$$

$$\nabla^2 T^* = \frac{\partial T^*}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial T^*}{\partial y}\frac{\partial \psi}{\partial x} + \sigma\frac{\partial T^*}{\partial t} \quad (4)$$

where $Da = K/L'^2$ is the Darcy number and $R = g\beta K L'^2 q'/k\alpha_f \nu$ is the Darcy–Rayleigh number, and K, g, β, and σ are the permeability of the porous material, the acceleration due to gravity, the coefficient of thermal expansion and the heat capacity ratio, respectively.

Due to the thermal boundary conditions considered here the resulting natural convection heat transfer is clearly transient. However, if the heating process is maintained long enough, a quasi-steady state will be reached, such that local temperature gradients, velocities and other parameters become nearly independent of time. At quasi-steady state, the temperature itself continues to increase with time, however its time dependence is the same at every point. Defining $T = T^* - St/\sigma$, equation (4) becomes at quasi-steady state

$$\nabla^2 T = \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial \psi}{\partial x} + S \quad (5)$$

and is now independent of time. In equation (5), $S = 1$ can be regarded as uniform sink term.

An approximate solution to the above problem can be sought in the case of a long shallow cavity ($A \to \infty$, where $A = H'/L'$ is the cavity aspect ratio). In this limit, as discussed in detail by Cormack *et al.* [17], Walker and Homsy [18], Vasseur *et al.* [16] and other authors, the flow in the central part of the cavity can be assumed to be parallel and in the x direction. Consequently, as demonstrated by Cormack *et al.* [17], the flow and temperature fields must be of the following form

$$\psi(x, y) = \psi(y) \quad (6)$$

and

$$T(x, y) = Cx + \theta(y) \quad (7)$$

where C is the unknown but constant temperature gradient in the x-direction. Substituting equations (6) and (7) into equations (3) and (5) one obtains respectively

$$\frac{d^4\psi}{dy^4} - \alpha^2\frac{d^2\psi}{dy^2} = \alpha^2 RC \quad (8)$$

and

$$C\frac{d\psi}{dy} = -\frac{d^2\theta}{dy^2} + S \quad (9)$$

where $\alpha^2 = Da^{-1}$.

The solutions to the above equations are of the form

$$\psi = RCF(y) \quad (10)$$

and

$$\theta = RC^2 Z(y) + \theta_c \quad (11)$$

where $F(y)$ and $Z(y)$ depend upon the hydrodynamic and thermal boundary conditions, imposed on the porous layer in the y-direction, respectively. θ_c is the temperature profile for pure conduction regime.

The thermal boundary conditions in the x-direction cannot be satisfied exactly with the parallel flow approximation. However, we can impose an equivalent energy flux condition in that direction (Bejan [19]) such that

$$C = \int_0^1 \left.\frac{d\psi}{dy}\right|_x \theta \, dy. \quad (12)$$

Substituting equations (10) and (11) into equation (12) it is readily found that

$$C = RCI_1 + R^2C^3I_2 \quad (13a)$$

or dividing by C

$$C = \pm\sqrt{((1 - RI_1)/I_2))}/R \quad (13b)$$

where the values of I_1 and I_2 are given respectively by

$$I_1 = \int_0^1 \frac{dF}{dy}\theta_c \, dy \quad (14)$$

$$I_2 = \int_0^1 \frac{dF}{dy} Z \, dy. \quad (15)$$

Thus the value of the axial temperature gradient C may be evaluated, from equation (13), for a given Darcy–Rayleigh number R and Darcy number Da.

From physical considerations, I_1 and I_2 are always positive and negative respectively. If $RI_1 > 1$, equation (13b) indicates that, in addition to the trivial solution $C = 0$ of equation (13a), two sets of solutions with positive and negative real roots of C exist, giving rise to convection cells in opposite directions. When $RI_1 < 1$, $C = 0$ is the only real root of (13a) and there is no convection. The marginal state, which determines the critical Rayleigh number R_c for the onset of convection is when $R_c I_1 = 1$, that is

$$R_c = 1/I_1. \quad (16)$$

This prediction of the critical Darcy–Rayleigh number is correctly obtained from the present parallel-flow analysis because the convection, that occurs when a constant heat flux is applied on the boundaries of a horizontal layer, is at zero wave number (Nield [10, 12], Kulacki and Goldstein [20]).

The Nusselt number Nu, for the present problem, is given by

$$Nu = \Delta T_c/\Delta T \quad (17)$$

where $\Delta T = T(0, 0) - T(0, 1)$ is the wall-to-wall dimensionless temperature differences and $\Delta T_c = 1/2$ the corresponding value for pure conduction regime.

As stated earlier, in the remaining part of this section we will consider situations where the bounding horizontal walls are both rigid or both stress-free or lower boundary rigid and upper boundary stress-free.

i) *Both horizontal boundaries rigid*

In this case the appropriate boundary conditions are

$$\psi = \mathrm{d}\psi/\mathrm{d}y = 0 \quad \mathrm{d}\theta/\mathrm{d}y = -1 \quad \text{at } y = 0$$
$$\psi = \mathrm{d}\psi/\mathrm{d}y = 0 \quad \mathrm{d}\theta/\mathrm{d}y = 0 \quad \text{at } y = 1. \tag{18}$$

Thus, as opposed to the Darcy model, the above boundary conditions imply a zero velocity on the rigid boundaries for any Da. The solutions to equations (8) and (9) are given by equations (10) and (11) where

$$F = \frac{1}{2\alpha}\left[\frac{\cosh\alpha\,(1/2-y)}{\sinh\alpha/2} - \coth\alpha/2 - \alpha(y^2-y)\right] \tag{19}$$

$$Z = -\frac{1}{2}\left[\frac{\sinh\alpha(1/2-y)}{\alpha^2\sinh\alpha/2} + \frac{y}{\alpha}\coth\alpha/2 + \frac{y^3}{3} - \frac{y^2}{2} - \frac{1}{\alpha^2}\right] \tag{20}$$

$$\theta_c = \frac{y^2}{2} - y. \tag{21}$$

Substituting equations (19)–(21) into equations (14) and (15) and integrating it is readily found that

$$I_1 = \frac{1}{24}\left[1 + \frac{12}{\alpha^2} - \frac{6}{\alpha}\coth\alpha/2\right] \tag{22}$$

and

$$I_2 = \frac{1}{4}\left[\frac{(\sinh\alpha-\alpha)}{2\alpha^3\sinh^2\alpha/2} - \frac{1}{30} + \frac{8}{\alpha^4} + \frac{\coth\alpha/2}{\alpha^3}\left(\frac{\alpha^2}{3} - 2 - \alpha\coth\alpha/2\right)\right]. \tag{23}$$

Thus, for given values of R and Da, the temperature gradient C can be evaluated from equations (13), (22) and (23). The stream function and temperature fields are then known from equations (10), (11) and (19)–(21).

From equations (16) and (22) the critical Rayleigh number is given by

$$R_c = 24\left[1 + \frac{12}{\alpha^2} - \frac{6}{\alpha}\coth(\alpha/2)\right]^{-1}. \tag{24}$$

We note that when α is very large, i.e. considering only the Darcy resistance, we have $R_c = 24$. However, when $\alpha \to 0$ (i.e. viscous fluid case), $Ra_c = 1440$ where $Ra = R/Da$ is the Rayleigh number for a viscous fluid. The equivalent problem of a fluid layer, heated internally by a uniform volumetric energy source ($S = -1$) and cooled from above by a constant heat flux while the lower boundary is insulated, has been studied by Kulacki and Goldstein [20]. Using linear and energy theory stability criteria, it was found that the onset of convection occurs at zero wavenumber when $Ra_c = 1433.6$.

From the temperature distribution, the Nusselt number is given by equation (17) as

$$Nu = \left[1 - RC^2\left(\frac{1}{6} + \frac{2}{\alpha^2} - \frac{1}{\alpha}\coth\alpha/2\right)\right]^{-1}. \tag{25}$$

ii) *Both horizontal boundaries stress-free*

If both upper and lower surface are stress-free the boundary conditions are

$$\psi = \mathrm{d}^2\psi/\mathrm{d}y^2 = 0 \quad \mathrm{d}\theta/\mathrm{d}y = -1 \quad \text{at } y = 0$$
$$\psi = \mathrm{d}^2\psi/\mathrm{d}y^2 = 0 \quad \mathrm{d}\theta/\mathrm{d}y = 0 \quad \text{at } y = 1. \tag{26}$$

Here, the velocity gradient is zero at the boundary for any Da. Such a constraint does not exist in the Darcy model. The functions F and Z, satisfying the boundary conditions (26), are given by

$$F = \frac{1}{\alpha^2}\left[\left[\frac{\cosh\alpha\,(1/2-y)}{\cosh\alpha/2} - 1\right] - \frac{\alpha^2}{2}(y^2-y)\right] \tag{27}$$

$$Z = -\left[\frac{\sinh\alpha\,(1/2-y)}{\alpha^3\cosh\alpha/2} + \frac{y}{\alpha^2} + \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^2}{2}\right) - \frac{\tanh\alpha/2}{\alpha^3}\right]. \tag{28}$$

Substituting equations (27), (28), and (21) into equations (14) and (15) and integrating yields

$$I_1 = \frac{1}{24}\left[1 - \frac{12}{\alpha^2} + \frac{24}{\alpha^3}\tanh\alpha/2\right] \tag{29}$$

and

$$I_2 = \frac{(\sinh\alpha-\alpha)}{2\alpha^5\cosh^2\alpha/2} - \frac{1}{120} + \frac{1}{6\alpha^2} - \frac{3}{\alpha^4} + \frac{6}{\alpha^5}\tanh\alpha/2. \tag{30}$$

Thus, the temperature gradient C can be evaluated from equations (13), (29) and (30) and the stream function and temperature fields are then known from equations (10), (11), (27), (28), and (21).

The critical Rayleigh number, for the onset of convection, is obtained from equations (16) and (29) as

$$R_c = 24\left[1 - \frac{12}{\alpha^2} + \frac{24}{\alpha^3}\tanh\alpha/2\right]^{-1} \tag{31}$$

such that $R_c = 24$ when α is very large (Darcy case) and $Ra_c = 240$ when α is very small (viscous fluid case).

From the temperature distribution, the Nusselt number is

$$Nu = \left[1 - RC^2\left(\frac{2\tanh\alpha/2}{\alpha^3} - \frac{1}{\alpha^2} + \frac{1}{12}\right)\right]^{-1}. \tag{32}$$

iii) *Lower boundary rigid and upper stress-free*

In this case, the appropriate boundary conditions are

$$\psi = d\psi/dy = 0 \quad d\theta/dy = -1 \quad \text{at } y=0$$
$$\psi = d^2\psi/dy^2 = 0 \quad d\theta/dy = 0 \quad \text{at } y=1. \tag{33}$$

The functions F and Z satisfying the above boundary conditions are

$$F = \frac{1}{H\alpha}\left[A\sinh\alpha y + B(\cosh\alpha y - 1) + (y-y^2)\frac{\alpha H}{2} + D\alpha y\right] \tag{34}$$

$$Z = -\frac{1}{2H\alpha}\left[A + \alpha B y - \frac{\alpha^2}{2}\left(D+\frac{H}{2}\right)y^2 + \frac{\alpha^2 H}{6}y^3 - A\cosh\alpha y - B\sinh\alpha y\right] \tag{35}$$

where

$$A = \alpha\cosh\alpha - (4/\alpha)\sinh^2\alpha/2$$
$$B = (2/\alpha)\sinh\alpha - \alpha\sinh\alpha - 2$$
$$D = -\sinh\alpha + (4/\alpha)\sinh^2\alpha/2$$
$$H = 2(\sinh\alpha - \alpha\cosh\alpha). \tag{36}$$

The values of I_1 and I_2 are found from equations (14), (15), (21) and (34)–(36). For this situation, the critical Rayleigh number is given by:

$$Rc = 24\left[1 + \frac{16\cosh^2\alpha/2}{H\alpha}(\alpha\tanh\alpha/2 - 2\tanh^2\alpha/2)\right]^{-1} \tag{37}$$

which has a value $Ra_c = 720$ when $\alpha \to 0$. Using linear and energy theory stability criteria, a value $Ra_c = 723.2$ was obtained by Kulacki and Goldstein [20] for a fluid layer.

The Nusselt number is given by equations (17), (21) and (35) as:

$$Nu = \left[1 - \frac{RC^2}{\alpha H}\left[-2 + \sinh\alpha\left(\frac{6}{\alpha} - \frac{12}{\alpha^2} - \frac{\alpha}{3}\right) + \cosh\alpha\left(\frac{\alpha^2}{3} - \frac{4}{\alpha^2}\right) + 4\frac{\cosh^2\alpha}{\alpha^2}\right]\right]^{-1}. \tag{38}$$

With the present hydrodynamic boundary conditions, it is also of interest to consider the case of a layer heated by uniformly distributed heat source ($S = -1$) and cooled from above by a constant heat flux. This problem is equivalent to thermal boundary conditions (33) with stress-free and rigid hydrodynamic boundary conditions on lower and upper boundaries respectively. For this situation, $\theta_c = -y^2/2$ and from equations (14), (16) and (34) it is found that:

$$R_c = 24\left[1 - \frac{12}{H\alpha^3}[AF_1 + BF_2 + D\alpha^3/3]\right]^{-1} \tag{39}$$

where A, B, D and H are given by equations (36) and

$$F_1 = (\alpha^2+2)\sinh\alpha - 2\alpha\cosh\alpha$$
$$F_2 = (\alpha^2+2)\cosh\alpha - 2\alpha\sinh\alpha - 2.$$

From equation (39) it results that $Ra_c = 576$ when $\alpha \to 0$, a value already obtained by Kulacki and Goldstein [20] for a fluid layer.

CONVECTION DUE TO SUDDEN HEATING OR COOLING

In this section we will show that the parallel flow analysis, in contrast with the one-term Galerkin approximation used by Rudraiah *et al.* [15], provides a smooth transition between the pure fluid layer and the Darcy porous medium, in the establishment of the critical Rayleigh for the onset of convection. Both the piecewise linear and step function temperature profiles studied by Rudraiah *et al.* [15] will be considered. The governing equations are still given by equations (3) and (5) with the uniform sink term S set to zero. We recall here that the parallel flow analysis lies on the established fact that the onset of convection occurs at zero wavenumber when constant heat flux conditions are applied on upper and lower boundaries, as demonstrated formally by Nield [12], both in the case of a fluid and a Darcy horizontal layer and also by Rudraiah *et al.* [15] for a Brinkman layer. Under this condition, it will be shown that the parallel flow analysis leads to a closed-form solution of R_c as a function of Da for all the cases considered in this section.

i) *Step function temperature profile*

We consider here the step-function temperature profile, in which the basic temperature drops suddenly by an amount ΔT at $y = \eta$, such that

$$\theta_c = \begin{cases} 0, & 0 \leqslant y < \eta \\ -1, & \eta < y \leqslant 1. \end{cases} \tag{40}$$

When both boundaries are rigid (RR), the critical Rayleigh number is obtained from equations (14), (16), (19) and (40) as

$$R_c = 2\left[\eta(1-\eta) + \frac{1}{\alpha}\left[\frac{\cosh\alpha(1/2-\eta)}{\sinh\alpha/2} - \coth\alpha/2\right]\right]^{-1}. \tag{41}$$

As η increases from 0 to 1, R_c decreases from ∞ to a minimum value and then increases again to ∞. The minimum value of R_c, attained at $\eta = 0.5$, i.e. midway between the boundaries, is given by:

$$R_c = \left[\frac{1}{8} + \frac{(1-\cosh\alpha/2)}{2\alpha\sinh\alpha/2}\right]^{-1} \tag{42}$$

such that we get $R_c = 8$ when $\alpha \to \infty$ and $Ra_c = 384$ when $\alpha \to 0$.

When both boundaries are stress-free (FF), equations (14), (16), (27) and (40) yield:

$$R_c = \left[\frac{\eta(1-\eta)}{2} - \frac{1}{\alpha^2}\left[1 - \frac{\cosh\alpha(1/2-\eta)}{\cosh\alpha/2}\right]\right]^{-1}. \tag{43}$$

This has a minimum value:

$$R_c = \left[\frac{1}{8} - \frac{1}{\alpha^2}\left[1 - \frac{1}{\cosh\alpha/2}\right]\right]^{-1} \tag{44}$$

when $\eta = 0.5$. When $\alpha \to 0$, $Ra_c = 384/5 = 76.8$ which is the known value (Nield [12]).

In the case when the lower boundary is rigid and the upper stress-free (RF), it is found from equations (14), (16), (34) and (40) that

$$R_c = \left[\frac{\eta}{2}(\eta-1) - \frac{1}{\alpha^2} - \left[\frac{F+(1-\eta)D}{H}\right]\right]^{-1} \tag{45}$$

where

$$F = \sinh\alpha(1-\eta) + \frac{2}{\alpha}\cosh\alpha\eta - \frac{4}{\alpha^2}\cosh\alpha(1/2-\eta)\sinh\alpha/2$$

and H and D are given in equations (36).

When $\alpha \to \infty$ equation (45) has a minimum value $R_c = 8$ attained at $\eta = 0.5$. For small values of α, $Ra_c = 184.6$ is minimum when $\eta = 0.578$. Thus, as the Darcy number is increased, the influence of the upper free surface becomes significant and the most destabilizing stepfunction has the step closer and closer to the free surface.

The critical Darcy–Rayleigh number R_c for the onset of convection in a Brinkman layer with a step-function temperature profile is presented in Fig. 1 as a function of Da for the three hydrodynamic boundary conditions considered in this section. The solid curves are the results of equations (42), (44) and the minimum values of equation (45) which are obtained at a particular η (η_m) function of Da. As mentioned earlier, the values $R_c = 8$ for a Darcy medium and $Ra_c = 384$ and 184.6 for a fluid layer with RR and RF boundaries have been predicted in the past by Nield [12]. When Da is small ($\leqslant 10^{-5}$), the three curves predicted by the present study are seen to approach asymptotically the Darcy value $R_c = 8$. Also, when Da is large enough, each of the curves tends asymptotically toward the particular critical Rayleigh number for a fluid layer with corresponding hydrodynamic boundary conditions. Also shown in Fig. 1 are the results predicted by Rudraiah *et al.* [15] on the basis of a Galerkin approximation. Although their results predict the viscous fluid situation correctly, they are obviously wrong in the limit of a Darcy situation ($Da \to 0$). The failure to predict correctly the Darcy limit encountered in the work by Rudraiah *et al.* [15] is due to the poor approximation (one-term Galerkin expansion) used to solve the Brinkman equation. It must be noticed though that a similar approximation applied separately to the Darcy and to the fluid equations provides accurate values of the critical Rayleigh numbers (Nield [12]).

For the case where the lower and upper boundaries are respectively rigid and stress-free (RF), it has already been mentioned that η_m is a function of Da. The asymptotic values $\eta_m = 0.5$ and $\eta_m = 0.578$ predicted by Nield for the Darcy and the pure fluid limit respectively are recovered by the present solution whereas the solution by Rudraiah *et al.* [15] predicts a constant value $\eta_m = 0.5$ for any Da.

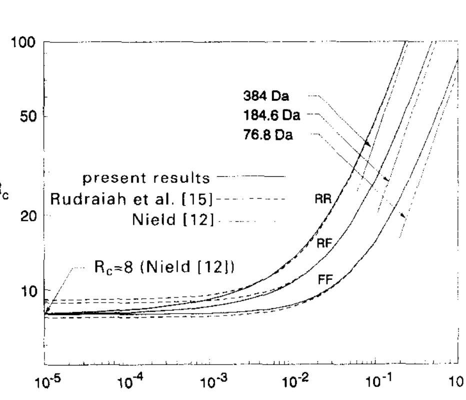

FIG. 1. Critical Darcy–Rayleigh number R_c as a function of Darcy number Da for a step function temperature profile.

ii) *Piecewise-linear profile*

We consider now the piecewise-linear profile, which approximates the profile for heating from below, given by

$$\theta_c = \begin{bmatrix} -y/\eta, & 0 \leqslant y < \eta \\ -1, & \eta < y \leqslant 1. \end{bmatrix} \tag{46}$$

When both the boundaries are rigid, it may be shown, from equations (14), (16), (19) and (46), that the critical Rayleigh number is given by

$$R_c = \left[\frac{(3\eta-2\eta^2)}{12} + \frac{1}{2\eta\alpha^2}\left[1 - \frac{\sinh\alpha(1/2-\eta)}{\sinh\alpha/2} - \frac{\coth\alpha/2}{\alpha}\right]^{-1} \tag{47}$$

which has a minimum of 96/9, attained at $\eta = 0.75$, when $\alpha \to \infty$ and a minimum of 601.1, at $\eta = 0.724$, when $\alpha \to 0$.

The critical Rayleigh number for a Brinkman layer heated from below and cooled from above by constant heat fluxes is obtained, by substituting $\eta = 1$ into equation (47), as

$$R_c = \left[\frac{1}{12} + \frac{1}{\alpha^2} - \frac{\coth \alpha/2}{\alpha}\right]^{-1} \tag{48}$$

yielding the exact values $R_c = 12$ when $\alpha \to \infty$ and $Ra_c = 720$ when $\alpha \to 0$.

When both the boundaries are stress-free, it is readily found, from equations (14), (16), (27) and (46), that:

$$R_c = \left[\frac{(3\eta - 2\eta^2)}{12} - \frac{1}{\alpha^2} + \frac{1}{\eta\alpha^3}\left[\tanh \alpha/2 - \frac{\sinh \alpha(1/2-\eta)}{\cosh \alpha/2}\right]\right]^{-1} \tag{49}$$

which has a minimum value of 96/9, attained at $\eta = 0.75$, when $\alpha \to \infty$ and a minimum value of $Ra_c = 105.57$, at $\eta = 0.744$, when $\alpha \to 0$.

When $\eta = 1$, equation (49) reduces to:

$$R_c = \left[\frac{1}{12} - \frac{1}{\alpha^2} + \frac{2}{\alpha^3}\tanh \alpha/2\right]^{-1} \tag{50}$$

such that $R_c = 12$ when $\alpha \to \infty$ and $Ra_c = 120$ when $\alpha \to 0$, the known values for a Darcy and a fluid layer heated from below by a basic temperature gradient $\partial\theta_c/\partial z = 1$.

When the lower boundary is rigid and the upper stress-free, equations (14), (16), (34) and (46) yield:

$$R_c = \left[\frac{(3\eta - 2\eta^2)}{12} + \frac{AE + BF + D\eta\alpha^2(\eta/2 - 1)}{H\eta\alpha^2}\right]^{-1} \tag{51}$$

where

$$E = \cosh \eta\alpha - \eta\alpha \sinh \alpha - 1$$

$$F = \sinh \eta\alpha - \eta\alpha \cosh \alpha$$

and A, B, D and H are given in equations (36). The above equation has a minimum of $R_c = 96/9$, attained at $\eta = 0.75$, when $\alpha \to \infty$ while $Ra_c = 292.6$, attained at $\eta = 0.821$, when $\alpha \to 0$.

Setting $\eta = 1$ into equation (51) it is found that

$$R_c = \left[\frac{1}{12} + \frac{\cosh^2 \alpha/2}{\alpha^2 H}[2\alpha + (\alpha^2 - 8)\tanh \alpha/2 - (\alpha - 4/\alpha)\tanh^2 \alpha/2]\right]^{-1} \tag{52}$$

such that $R_c = 12$ when $\alpha \to \infty$ and $Ra_c = 320$ when $\alpha \to 0$, the known values for a Darcy and a fluid layer heated from below and cooled from above by constant heat fluxes.

For the piecewise linear profile given by

$$\theta_c = \begin{cases} 0, & 0 \leqslant y < \eta \\ (1-y)/(1-\eta), & \eta < y \leqslant 1 \end{cases} \tag{53}$$

which approximates the profile for cooling from above. For this situation, when the upper boundary is stress-free and the lower rigid, we have

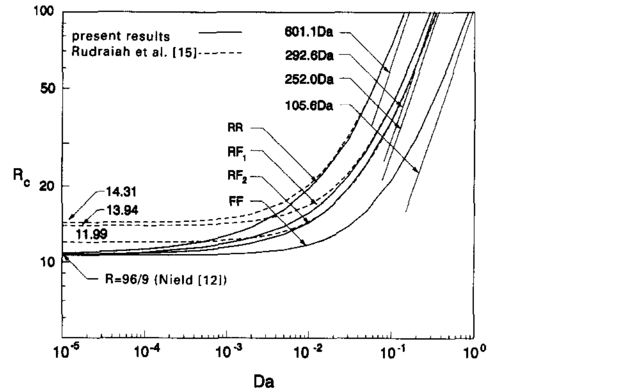

FIG. 2. Critical Darcy–Rayleigh number R_c as a function of Darcy number for a piecewise-linear temperature profile.

$$R_c\left[\frac{(1 - 3\eta^2 + 2\eta^3)}{12(1-\eta)} - \frac{[AE + BF + D\alpha^2(1+\eta^2)/2]}{H\alpha^2(1-\eta)}\right]^{-1} \tag{54}$$

where

$$E = \alpha \sinh \alpha(1-\eta) - \cosh \alpha + \cosh \alpha\eta$$

$$F = \alpha \cosh \alpha(1-\eta) - \sinh \alpha + \sinh \alpha\eta.$$

Equation (54) has a minimum value of $R_c = 96/9$, attained at $\eta = 0.25$, when $\alpha \to \infty$ and a minimum value of $Ra_c = 252$, attained at $\eta = 0.362$, when $\alpha \to 0$.

The minimum value of R_c corresponding to a specific η (η_m), has been plotted as a function of Da for each of the equations (47), (49), (51) and (54) in Fig. 2 (curves RR, FF, RF_1 and RF_2, respectively). η_m as a function of Da is shown in Figs. 3(a), (b).

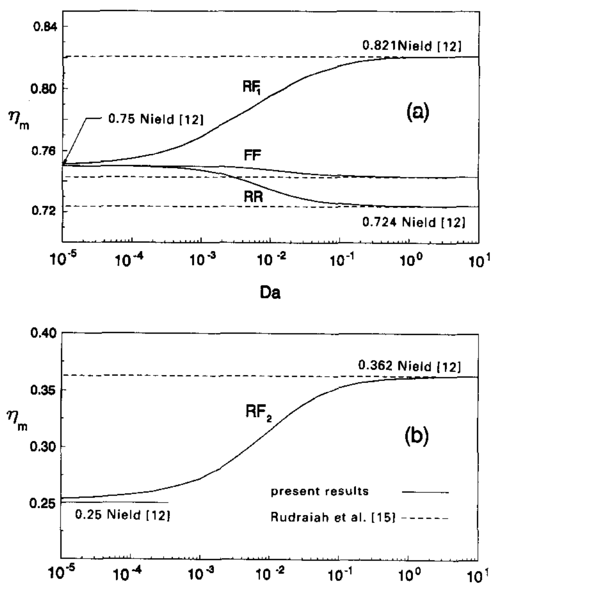

FIG. 3. Thermal depth parameter η_m as a function of Darcy number for the onset of convection for a piecewise-linear temperature profile: (a) RR, FF, RF_1; (b) RF_2.

CONCLUSIONS

The problem of unicellular laminar natural convection in a horizontal porous layer heated from below by a constant heat flux, while the other boundaries are maintained adiabatic, has been investigated analytically, using the Brinkman model. Although the problem is basically transient, it is shown that, at a sufficiently large time after heating, a quasi-steady state is reached for which local temperature gradients, velocities and other parameters become very nearly independent of time. The governing equations for the porous layer are solved analytically, in the limit of a thin layer using various combinations of hydrodynamic boundary conditions at the upper and lower surfaces. Results are obtained in terms of an overall Nusselt number as a function of Rayleigh and Darcy numbers. The critical Rayleigh number for the onset of convection is predicted explicitly in terms of the Darcy number for each of the hydrodynamic boundary conditions considered in this study.

The effect of nonlinear temperature distribution, arising either from sudden heating or cooling in a fluid saturated porous medium, has also been considered. Using the parallel flow analysis, 'exact' values for the critical Rayleigh number have been obtained for various combinations of hydrodynamic boundary conditions at the upper and lower adiabatic surfaces. The results obtained are compared to the existing values for a viscous fluid ($Da \rightarrow \infty$) and the Darcy porous medium ($Da \rightarrow 0$) and good agreement is found. The same problem has been solved in the past by Rudraiah *et al.* [15] using a single-term Galerkin approximation but their results showed a poor agreement with the solution of Nield [12] in the Darcy limit. It is demonstrated that this is due, as suggested by Nield [21, 22], to the crude approximation used by Rudraiah *et al.* [15] in their calculations.

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